8. $I = \int \frac{dx}{(2+\cos x)\sin x}$

$R(\sin x, \cos x) = -R(-\sin x, \cos x) \Rightarrow$ substituce $u = \cos x$

$du = -\sin x \, dx$;

$$I = \int \frac{\sin x \, dx}{(2+\cos x)\underbrace{\sin^2 x}_{1-\cos^2 x}} = -\int \frac{du}{(2+u)(1-u^2)} = \int \frac{du}{(u+2)(u^2-1)} = \int \frac{du}{(u+2)(u-1)(u+1)}$$

$$\frac{1}{(u+2)(u-1)(u+1)} = \frac{a}{u+2} + \frac{b}{u-1} + \frac{c}{u+1}$$

$u \to -2: \quad a = \frac{1}{(-2-1)\cdot(-2+1)} = \frac{1}{3}$

$u \to 1: \quad b = \frac{1}{(1+2)(1+1)} = \frac{1}{6}$

$u \to -1: \quad c = \frac{1}{(-1+2)(-1-1)} = \frac{-1}{2}$

$$I = \int \left(\frac{\frac{1}{3}}{u+2} + \frac{\frac{1}{6}}{u-1} + \frac{-\frac{1}{2}}{u+1} \right) du = \frac{1}{3}\log|u+2| + \frac{1}{6}\log|u-1| - \frac{1}{2}\log|u+1| + C$$

$$= \frac{1}{3}\log(2+\cos x) + \frac{1}{6}\log(1-\cos x) - \frac{1}{2}\log(1+\cos x) + C$$

Definiční obor: $\sin x \neq 0$ (tj. $\cos x \neq \pm 1$), $x \neq \pi k$, $k \in \mathbb{Z}$, DO: $\bigcup_{k\in\mathbb{Z}} (\pi k, \pi + \pi k)$

výsledek platí na každém z intervalu definičního oboru

<u>Výsledek</u>: $I = \frac{1}{3}\log(2+\cos x) + \frac{1}{6}\log(1-\cos x) - \frac{1}{2}\log(1+\cos x) + C_k$, $x \in (\pi k, \pi(k+1))$, $k \in \mathbb{Z}$

$$10.\ I=\int \frac{\sin x \cdot \cos x \, dx}{1+\sin^4 x}$$

$$R(\sin x, -\cos x) = -R(\sin x, \cos x) \implies \text{substituce } u=\sin x$$

$$du = \cos x \, dx$$

$$I = \int \frac{\sin x \overbrace{\cos x\, dx}^{du}}{1+\sin^4 x} = \int \frac{u\, du}{1+u^4} = \left| \begin{matrix} v=u^2 \\ dv = 2u\, du \end{matrix} \right| = \frac{1}{2}\int \frac{dv}{1+v^2} = \frac{1}{2}\operatorname{arctg} v + C$$

$$= \frac{1}{2}\operatorname{arctg}(u^2) + C = \frac{1}{2}\operatorname{arctg}(\sin^2 x) + C, \qquad \text{DO: } x \in \mathbb{R}$$

Výsledek: $I = \frac{1}{2}\operatorname{arctg}(\sin^2 x) + C,\ x \in \mathbb{R}$

12. $I=\int \frac{dx}{5+\cos x}$

nic jiného nefunguje, děláme univerzální substituce $\quad x\in\left(-\pi+2\pi k, \pi+2\pi k\right)$

$$t=\operatorname{tg}\frac{x}{2}, \quad \frac{x}{2}\in\left(-\frac{\pi}{2}+\pi k, \frac{\pi}{2}+\pi k\right), \ k\in\mathbb{Z} \rightarrow x\in\left(-\pi+2\pi k, \pi+2\pi k\right)$$

$$\cos x = \cos^2\frac{x}{2}-\sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{1-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{1+\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{1-t^2}{1+t^2}$$

$$dt = \frac{dx}{2\cos^2\frac{x}{2}} = \frac{\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)dx}{2\cos^2\frac{x}{2}} = \frac{1}{2}(1+t^2)dx; \qquad dx=\frac{2\,dt}{1+t^2}$$

teď nepotřebujeme, ale pro případ: $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} =$

$$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$I=\int\frac{dx}{5+\cos x} = \int\frac{2\,dt}{(1+t^2)\left(5+\frac{1-t^2}{1+t^2}\right)} = \left|\begin{array}{l} t=\sqrt{\frac{3}{2}}u \\ dt=\sqrt{\frac{3}{2}}du \end{array}\right| = \int\frac{\sqrt{\frac{3}{2}}du}{2\cdot\frac{3}{2}u^2+3} =$$

$$= \int\frac{2\,dt}{5(1+t^2)+1-t^2} = \int\frac{2dt}{4t^2+6} = \int\frac{dt}{2t^2+3}$$

chceme 1

$$= \int\frac{\sqrt{\frac{3}{2}}du}{3(u^2+1)} = \frac{1}{\sqrt{6}}\operatorname{arctg}u + C = \frac{1}{\sqrt{6}}\operatorname{arctg}\left(\sqrt{\frac{2}{3}}t\right)+C = \frac{1}{\sqrt{6}}\operatorname{arctg}\left(\sqrt{\frac{2}{3}}\operatorname{tg}\frac{x}{2}\right)+C$$

Tak že, pro $x\in\left(-\pi+2\pi k, \pi+2\pi k\right), k\in\mathbb{Z}$, máme $I=\frac{1}{\sqrt{6}}\operatorname{arctg}\left(\sqrt{\frac{2}{3}}\operatorname{tg}\frac{x}{2}\right)+C_k$

Pro $x\rightarrow \pi+2\pi k-0$: $\quad \operatorname{tg}\frac{x}{2}\rightarrow+\infty, \quad I\rightarrow\frac{1}{\sqrt{6}}\cdot\frac{\pi}{2}+C_k$

Pro $x\rightarrow \pi+2\pi k+0 = -\pi+2\pi(k+1)+0$: $\quad \operatorname{tg}\frac{x}{2}\rightarrow-\infty, \quad I\rightarrow\frac{-1}{\sqrt{6}}\cdot\frac{\pi}{2}+C_{k+1}$

Vezmeme ("lepíme") $\quad \frac{1}{\sqrt{6}}\cdot\frac{\pi}{2}+C_k = -\frac{1}{\sqrt{6}}\cdot\frac{\pi}{2}+C_{k+1};$

$$C_{k+1} = C_k - \frac{\pi}{\sqrt{6}}$$

indukce: vyjádříme všechny C_k přes C_0:

$$C_k = C_0 - \frac{\pi k}{\sqrt{6}}, \quad k\in\mathbb{Z}$$

Výsledek:

$$I=\frac{1}{\sqrt{6}}\operatorname{arctg}\left(\sqrt{\frac{2}{3}}\operatorname{tg}\frac{x}{2}\right)-\frac{\pi k}{\sqrt{6}}+C, \quad x\in\left[-\pi+2\pi k, \pi+2\pi k\right), \ k\in\mathbb{Z}$$

13. $I = \int \frac{dx}{(\sin^2 x + 2\cos^2 x)^2}$

$R(\sin x, \cos x) = R(-\sin x, -\cos x) \Longrightarrow$ substituce $t = \operatorname{tg} x$

$x \in \left(-\frac{\pi}{2}+\pi k, \frac{\pi}{2}+\pi k\right)$, $k \in \mathbb{Z}$

$$dt = \frac{1}{\cos^2 x}\, dx = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\, dx = (1+\operatorname{tg}^2 x)\, dx = (1+t^2)\, dx$$

$$dx = \frac{dt}{1+t^2}$$

$$\sin^2 x + 2\cos^2 x = \cos^2 x(\operatorname{tg}^2 x + 2) = \frac{\cos^2 x}{\cos^2 x + \sin^2 x}\cdot(\operatorname{tg}^2 x+2) = \frac{\operatorname{tg}^2 x + 2}{1+\operatorname{tg}^2 x}$$

$$I = \int \frac{dt}{(1+t^2)\left(\frac{t^2+2}{t^2+1}\right)^2} = \int \frac{t^2+1}{\underbrace{(t^2+2)^2}_{\text{chceme } 1}}\,dt = \left| \begin{matrix} t=\sqrt{2}\, u \\ dt = \sqrt{2}\, du\end{matrix}\right| = \int \frac{(2u^2+1)\sqrt{2}\, du}{(2u^2+2)^2} = \frac{\sqrt{2}}{4}\int\frac{(2u^2+1)\,du}{(u^2+1)^2}$$

$$= \frac{\sqrt{2}}{2}\int\frac{(u^2+\frac12)\,du}{(u^2+1)^2} = \frac{\sqrt{2}}{2}\int\frac{(u^2+1-\frac12)\,du}{(u^2+1)^2} = \frac{\sqrt{2}}{2}\underbrace{\int\frac{du}{u^2+1}}_{\operatorname{arctg} u} - \frac{\sqrt{2}}{4}\underbrace{\int\frac{du}{(u^2+1)^2}}_{\text{per partes začínající z } \int\frac{du}{u^2+1}} =$$

$$\operatorname{arctg} u = \int \underbrace{u'}_{1} \cdot \frac{1}{u^2+1}\, du = \frac{u}{u^2+1} + \int\frac{u\cdot 2u\, du}{(u^2+1)^2} = \frac{u}{u^2+1} + 2\int\frac{(u^2+1-1)\,du}{(u^2+1)^2} = \frac{u}{u^2+1} + 2\int\frac{du}{u^2+1} - 2\int\frac{du}{(u^2+1)^2}$$

$$\Rightarrow \boxed{\int\frac{du}{(u^2+1)^2} = \frac12 \operatorname{arctg} u + \frac{u}{2(u^2+1)} + C} \quad u\in\mathbb{R}$$

kontrola: $\frac{d}{du}$: $\frac{1}{2(1+u^2)} + \frac{1}{2(u^2+1)} - \frac{u\cdot 2u}{2(u^2+1)^2} = \frac{1}{1+u^2} - \frac{u^2}{(u^2+1)^2} = \frac{1+u^2-u^2}{(u^2+1)^2}$ ✓

$$= I = \frac{\sqrt{2}}{2}\operatorname{arctg} u - \frac{\sqrt{2}}{4}\left(\frac12 \operatorname{arctg} u + \frac{u}{2(u^2+1)}\right) + C =$$

$$= \frac{3\sqrt{2}}{8}\operatorname{arctg} u - \frac{\sqrt{2}}{8}\,\frac{u}{u^2+1} + C = \left| u = \frac{t}{\sqrt{2}} = \frac{\operatorname{tg} x}{\sqrt{2}}\right|$$

$$= \frac{3\sqrt{2}}{8}\operatorname{arctg}\left(\frac{\operatorname{tg} x}{\sqrt{2}}\right) - \frac{\sqrt{2}}{8}\,\frac{\frac{\operatorname{tg} x}{\sqrt{2}}}{\frac{\operatorname{tg}^2 x}{2}+1} + C$$

$$= \frac{3\sqrt{2}}{8}\operatorname{arctg}\left(\frac{\operatorname{tg} x}{\sqrt{2}}\right) - \frac14\, \frac{\operatorname{tg} x}{\operatorname{tg}^2 x+2} + C_k, \quad x \in \left(-\frac{\pi}{2}+\pi k, \frac{\pi}{2}+\pi k\right),\ k\in\mathbb{Z}$$

"lepení": $x \to \frac{\pi}{2} + \pi k - 0$: $\operatorname{tg} x\to+\infty$: $I \to \frac{3\sqrt{2}}{8}\cdot\frac{\pi}{2} + C_k$

$x\to \frac{\pi}{2}+\pi k+0 = -\frac{\pi}{2}+\pi(k+1)+0$: $\operatorname{tg} x\to-\infty$: $I\to -\frac{3\sqrt{2}}{8}\cdot\frac{\pi}{2} + C_{k+1}$

Takže $-\frac{3\sqrt{2}}{8}\cdot\frac{\pi}{2} + C_{k+1} = \frac{3\sqrt{2}}{8}\cdot\frac{\pi}{2} + C_k$; $\quad C_{k+1} = \frac{3\sqrt{2}}{8}\cdot\pi + C_k$

Pak $C_k = \frac{3\sqrt{2}}{8}\cdot\pi k + C_0$

Výsledek:

$$I = \begin{cases} \frac{3\sqrt{2}}{8}\operatorname{arctg}\left(\frac{\operatorname{tg} x}{\sqrt{2}}\right) - \frac14\,\frac{\operatorname{tg} x}{\operatorname{tg}^2 x+2} + \frac{3\sqrt{2}}{8}\pi k + C_0, & x\in\left(-\frac{\pi}{2}+\pi k, \frac{\pi}{2}+\pi k\right),\ k\in\mathbb{Z} \\ \frac{3\sqrt{2}}{8}\pi\left(k+\frac12\right) + C_0, & x = \frac{\pi}{2}+\pi k,\ k\in\mathbb{Z} \end{cases}$$

14. $I=\int \sqrt{\frac{x+1}{x-1}}\,\frac{dx}{x}$

$u=\sqrt{\frac{x+1}{x-1}}$ $\quad x\in(-\infty,-1)\cup(1,+\infty)$

$u^2=\frac{x+1}{x-1}; \quad (x-1)u^2=x+1$

$x(u^2-1)=u^2+1 \qquad x\in(1,+\infty)\to u\in(1,+\infty)$

$x=\frac{u^2+1}{u^2-1} \qquad x\in(-\infty,-1)\to u\in(0,1)$

$$x=\frac{u^2-1+2}{u^2-1}=1+\frac{2}{u^2-1}$$

$$dx=\frac{-2\cdot 2u\,du}{(u^2-1)^2}$$

$$I=\int\sqrt{\frac{x+1}{x-1}}\,\frac{dx}{x}$$

$$=\int u\cdot\frac{u^2-1}{u^2+1}\cdot\frac{-4u}{(u^2-1)^2}du=\int\frac{-4u^2\,du}{(u^2+1)(u^2-1)} \ominus$$

můžeme nejprve udělat rozklad na parciální zlomky pro ~~prom~~ u^2, a pak už pro u

$$\frac{-4u^2}{(u^2+1)(u^2-1)}=\frac{a}{u^2+1}+\frac{b}{u^2-1}$$

$$\left(\frac{-4v}{(v+1)(v-1)}=\frac{a}{v+1}+\frac{b}{v-1}\right)$$

$v\to-1:\ a=\frac{-4\cdot(-1)}{-1-1}=-2$

$v\to 1:\ b=\frac{-4\cdot 1}{1+1}=-2$

$$\frac{-4u^2}{(u^2+1)(u^2-1)}=\frac{-2}{u^2+1}+\frac{-2}{u^2-1}$$

$$\downarrow$$

$$\frac{-1}{u-1}+\frac{1}{u+1}$$

$$=\frac{-2}{u^2+1}+\frac{1}{u+1}-\frac{1}{u-1}$$

$$I\ominus\int\left(\frac{-2}{u^2+1}+\frac{1}{u+1}-\frac{1}{u-1}\right)du=-2\,\mathrm{arctg}\,u+\log|u+1|-\log|u-1|+C$$

$$=-2\,\mathrm{arctg}\,u+\log\left|\frac{u+1}{u-1}\right|+C=-2\,\mathrm{arctg}\sqrt{\frac{x+1}{x-1}}+\underbrace{\log\left|\frac{\sqrt{\frac{x+1}{x-1}}+1}{\sqrt{\frac{x+1}{x-1}}-1}\right|}+C$$

1) $x>1$: $\log\left|\frac{\sqrt{\frac{x+1}{x-1}}+1}{\sqrt{\frac{x+1}{x-1}}-1}\right|=\log\left|\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}\cdot\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\right|=\log\frac{(\sqrt{x+1}+\sqrt{x-1})^2}{(x+1)-(x-1)}$

$$=2\log(\sqrt{x+1}+\sqrt{x-1})-\log 2$$

2) $x<-1$: $\log\left|\frac{\sqrt{\frac{-1-x}{1-x}}+1}{\sqrt{\frac{-x-1}{-x+1}}-1}\right|=\log\left|\frac{\sqrt{-x-1}+\sqrt{-x+1}}{\sqrt{-x-1}-\sqrt{-x+1}}\cdot\frac{\sqrt{-x-1}+\sqrt{-x+1}}{\sqrt{-x-1}+\sqrt{-x+1}}\right|$

$$=\log\left|\frac{(\sqrt{-x-1}+\sqrt{-x+1})^2}{(-x-1)-(-x+1)}\right|=2\log(\sqrt{-x-1}+\sqrt{-x+1})-\log 2$$

Výsledek: $I=-2\,\mathrm{arctg}\sqrt{\frac{x+1}{x-1}}+2\log(\sqrt{|x+1|}+\sqrt{|x-1|})+C_1, \quad x>1$

$I=-2\,\mathrm{arctg}\sqrt{\frac{x+1}{x-1}}+2\log(\sqrt{|x+1|}+\sqrt{|x-1|})+C_2, \quad x<-1$

15. $\int \frac{x\,dx}{\sqrt{-x^2+2x+8}}$

$$x^2-2x-8=0 \qquad (x-1)^2=9; \qquad x-1=\pm 3, \quad \left[\begin{array}{l} x=4 \\ x=-2 \end{array}\right.$$

DO: $x \in (-2,4)$

$$\sqrt{-x^2+2x+8} = \sqrt{-(x+2)(x-4)} = (4-x)\sqrt{\frac{x+2}{4-x}}$$

substituce $u=\sqrt{\frac{2+x}{4-x}}$ $\qquad x\in(-2,4) \to u \in (0,+\infty)$

$$u^2=\frac{2+x}{4-x}; \quad 2+x=(4-x)u^2; \quad x(1+u^2)=4u^2-2$$

$$x=\frac{4u^2-2}{u^2+1} = \frac{4u^2+4-6}{u^2+1} = 1-\frac{6}{u^2+1} \qquad dx=\frac{12u\,du}{(u^2+1)^2}$$

$$I=\int\frac{x\,dx}{\sqrt{-x^2+2x+8}} = \int \frac{x}{(4-x)}\sqrt{\frac{4-x}{2+x}}\,dx=$$

$$=\int \frac{4u^2-2}{u^2+1}\cdot\frac{1}{\left(4-\frac{4u^2-2}{u^2+1}\right)}\cdot\frac{1}{u}\cdot\frac{12u\,du}{(u^2+1)^2}$$

$$=\int\frac{(4u^2-2)\cdot 12\,du}{(4(u^2+1)-4u^2+2)\cdot(u^2+1)^2} = \int\frac{(4u^2-2)\cdot 12\,du}{6\cdot(u^2+1)^2} = 4\int\frac{2u^2-1}{(u^2+1)^2}\,du$$

$$=4\int\frac{2u^2+2-3}{(u^2+1)^2}\,du = 8\int\frac{du}{u^2+1} - 12\int\frac{du}{(u^2+1)^2} \ominus$$

víme, z 13., že $\boxed{\int\frac{du}{(u^2+1)^2} = \frac{1}{2}\operatorname{arctg} u + \frac{u}{2(u^2+1)} + \check{C}}$

$$\ominus\ I = 8\operatorname{arctg} u - 12\left(\frac{1}{2}\operatorname{arctg} u + \frac{u}{2(u^2+1)} + \check{C}\right)$$

$$= 2\operatorname{arctg} u - \frac{6u}{u^2+1} + C$$

$$= 2\operatorname{arctg}\sqrt{\frac{2+x}{4-x}} - \frac{6\sqrt{\frac{2+x}{4-x}}}{\frac{2+x}{4-x}+1} + C = 2\operatorname{arctg}\sqrt{\frac{2+x}{4-x}} - \frac{6\sqrt{(2+x)(4-x)}}{2+x+4-x} + C$$

$$= 2\operatorname{arctg}\sqrt{\frac{2+x}{4-x}} - \sqrt{-x^2+2x+8} + C, \quad x\in(-2,4)$$

Kontrola:

$$\frac{2}{1+\frac{2+x}{4-x}}\cdot\sqrt{\frac{2+x}{4-x}}\cdot\frac{1}{2}\left(\frac{1}{2+x}+\frac{1}{4-x}\right) - \frac{-x+1}{\sqrt{-x^2+2x+8}} =$$

$$= \frac{2(4-x)}{6}\cdot\sqrt{\frac{2+x}{4-x}}\cdot\frac{1}{2}\cdot\frac{6}{(2+x)(4-x)} + \frac{x-1}{\sqrt{-x^2+2x+8}} = \sqrt{\frac{2+x}{4-x}}\cdot\frac{1}{(2+x)} + \frac{x-1}{\sqrt{-x^2+2x+8}} = \frac{x}{\sqrt{-x^2+2x+8}}$$

Výsledek:

$$I = \begin{cases} 2\operatorname{arctg}\sqrt{\frac{2+x}{4-x}} - \sqrt{-x^2+2x+8} + C, & x\in(-2,4) \\ \pi + C, & x=4 \\ C, & x=-2 \end{cases}$$

16 $\displaystyle\int \frac{\sqrt{x^2+x+1}}{x}\,dx$

$$\sqrt{x^2+x+1} = x+u$$

$$x^2+x+1 = x^2+2xu+u^2$$

$$x(1-2u) = u^2-1$$

$$x = \frac{u^2-1}{1-2u} = \frac{\left(u^2-\frac{u}{2}\right)+\left(\frac{u}{2}-\frac{1}{4}\right)-\frac{3}{4}}{1-2u}$$

$$= \frac{-u}{2} - \frac{1}{4} + \frac{3}{8\left(u-\frac{1}{2}\right)}$$

$$dx = \left[\frac{-1}{2} - \frac{3}{8\left(u-\frac{1}{2}\right)^2}\right]du$$

$$= \frac{4\left(u-\frac{1}{2}\right)^2+3}{-8\left(u-\frac{1}{2}\right)^2}\,du$$

$$= \frac{4\left(u^2-u+\frac{1}{4}\right)+3}{-8\left(u-\frac{1}{2}\right)^2}\,du$$

$$= \frac{(u^2-u+1)\,du}{-2\cdot\left(u-\frac{1}{2}\right)^2}$$

$$\sqrt{x^2+x+1} = x+u = \frac{u^2-1}{1-2u}+u = \frac{u^2-1+u-2u^2}{1-2u} = \frac{-u^2+u-1}{1-2u} = \frac{u^2-u+1}{2\left(u-\frac{1}{2}\right)} > 0 \quad \Rightarrow u > \tfrac{1}{2}$$

$$x \in (-\infty,+\infty) \longleftrightarrow u \in \left(\tfrac{1}{2},+\infty\right)$$

$$u = \sqrt{x^2+x+1}-x \qquad x\to-\infty \to u\to+\infty \qquad x\to+\infty \to u\to\tfrac{1}{2}$$

$$= \frac{x+1}{\sqrt{x^2+x+1}+x}$$

$$\int \frac{\sqrt{x^2+x+1}\,dx}{x} = \int \frac{u^2-u+1}{2\left(u-\frac{1}{2}\right)} \cdot \frac{-2\left(u-\frac{1}{2}\right)}{u^2-1} \cdot \frac{(u^2-u+1)\,du}{-2\left(u-\frac{1}{2}\right)^2} = \frac{1}{2}\int \frac{(u^2-u+1)^2\,du}{(u^2-1)\left(u-\frac{1}{2}\right)^2}$$

$$P = (u^2-u+1)^2 = u^4-2u^3+3u^2-2u+1$$

$$Q = (u^2-1)\left(u-\tfrac{1}{2}\right)^2 = (u^2-1)\left(u^2-u+\tfrac{1}{4}\right) = u^4-u^3-\tfrac{3}{4}u^2+u-\tfrac{1}{4}$$

$4 = \deg P \geq \deg Q = 4 \Rightarrow$ vydělíme se zbytkem

$$\begin{array}{rl} u^4-2u^3+3u^2-2u+1 & \Big|\; u^4-u^3-\frac{3}{4}u^2+u-\frac{1}{4} \\ -\;\underline{u^4-u^3-\frac{3}{4}u^2+u-\frac{1}{4}} & \;\;1 \\ -u^3+\frac{15}{4}u^2-3u+\frac{5}{4} & \end{array}$$

$$I = \frac{1}{2}\int\left[1+\frac{-u^3+\frac{15}{4}u^2-3u+\frac{5}{4}}{(u-1)(u+1)\left(u-\frac{1}{2}\right)^2}\right]du$$

$$\frac{-u^3+\frac{15}{4}u^2-3u+\frac{5}{4}}{(u-1)(u+1)\left(u-\frac{1}{2}\right)^2} = \frac{a}{u-1}+\frac{b}{u+1}+\frac{c}{\left(u-\frac{1}{2}\right)^2}+\frac{d}{u-\frac{1}{2}}$$

$$u\to1:\ a = \frac{-1+\frac{15}{4}-3+\frac{5}{4}}{2\cdot\frac{1}{4}} = \frac{1}{\frac{1}{2}} = 2; \qquad u\to-1:\ b = \frac{1+\frac{15}{4}+3+\frac{5}{4}}{-2\cdot\frac{9}{4}} = \frac{9}{-\frac{9}{2}} = -2$$

$u \to \frac{1}{2}$: $\quad c = \dfrac{-\frac{1}{8} + \frac{15}{16} - \frac{3}{2} + \frac{5}{4}}{-\frac{1}{2}\cdot\frac{3}{2}} = \dfrac{-2+15-24+20}{-3\cdot 4} = \dfrac{9}{-3\cdot 4} = \dfrac{-3}{4}$

aby najít d, vezmeme nějaký "jednoduchý" u, třeba $u=0$

$u \to 0$: $\quad \dfrac{\frac{5}{4}}{-\frac{1}{4}} = -a + b + 4c - 2d$

$-5 = -2 - 2 - 3 - 2d$; $\quad d = -1$

$$I = \int \left(\frac{1}{2} + \frac{1}{u-1} - \frac{1}{u+1} - \frac{\frac{3}{8}}{(u-\frac{1}{2})^2} - \frac{\frac{1}{2}}{u-\frac{1}{2}} \right) du$$

$$= \frac{u}{2} + \log|u-1| - \log(u+1) + \frac{3}{8(u-\frac{1}{2})} - \frac{1}{2}\log(u-\tfrac{1}{2}) + C$$

- $u - \frac{1}{2} = \sqrt{x^2+x+1} - x - \frac{1}{2}$

  $\dfrac{1}{u-\frac{1}{2}} = \dfrac{1}{\sqrt{x^2+x+1} - x - \frac{1}{2}} \cdot \dfrac{\sqrt{x^2+x+1} + x + \frac{1}{2}}{\sqrt{x^2+x+1} + x + \frac{1}{2}} = \dfrac{\sqrt{x^2+x+1} + x + \frac{1}{2}}{(x^2+x+1) - (x+\frac{1}{2})^2} =$

  $= \frac{4}{3}\left(\sqrt{x^2+x+1} + x + \frac{1}{2}\right)$

  $\dfrac{3}{8(u-\frac{1}{2})} = \frac{1}{2}\left(\sqrt{x^2+x+1} + x + \frac{1}{2}\right)$

- $\dfrac{u}{2} = \dfrac{\sqrt{x^2+x+1} - x}{2}$

- $\dfrac{u}{2} + \dfrac{3}{8(u-\frac{1}{2})} = \sqrt{x^2+x+1} + \frac{1}{4}$

- $\dfrac{u-1}{u+1} = \dfrac{\sqrt{x^2+x+1} - x - 1}{\sqrt{x^2+x+1} - x + 1} \cdot \dfrac{\sqrt{x^2+x+1} + x - 1}{\sqrt{x^2+x+1} + x - 1} = \dfrac{x^2+x+1 - x^2 + 1 + \sqrt{x^2+x+1}\,(-2)}{(x^2+x+1) - (x-1)^2}$

  $= \dfrac{x + 2 - 2\sqrt{x^2+x+1}}{3x} = \dfrac{(x+2)^2 - 4(x^2+x+1)}{3x(x+2+2\sqrt{x^2+x+1})} = \dfrac{-3x^2}{3x(x+2+2\sqrt{x^2+x+1})} = \dfrac{-x}{x+2+2\sqrt{x^2+x+1}}$

- $\cancel{\dfrac{(u-1)^2}{(u+1)^2 (u-\frac{1}{2})}} = \cancel{\dfrac{(x+2-2\sqrt{x^2+x+1})^2}{9x^2} \cdot \frac{4}{3}\left(x + \frac{1}{2} + \sqrt{x^2+x+1}\right)}$

  $\cancel{\dfrac{(u-1)^2}{(u+1)^2} = \dfrac{1}{9x^2}\left[x^2 + 4x + 4\right.}$

$$I = \sqrt{x^2+x+1} + \log \frac{|x|}{x+2+2\sqrt{x^2+x+1}} + \frac{1}{2}\log\left(x + \frac{1}{2} + \sqrt{x^2+x+1}\right) + C,$$

$x \in (-\infty, 0)$: $C = C_1$

$x \in (0, +\infty)$: $C = C_2$

Kontrola

1) $\dfrac{x+\frac{1}{2}}{\sqrt{x^2+x+1}}$

2) $\dfrac{1}{x}$

3) $$\frac{-1\cdot\left(1+\frac{2(x+\frac{1}{2})}{\sqrt{x^2+x+1}}\right)}{x+2+2\sqrt{x^2+x+1}} = \frac{2x+1+\sqrt{x^2+x+1}}{-\sqrt{x^2+x+1}\left(x+2+2\sqrt{x^2+x+1}\right)} \cdot =$$

$$= -\frac{\left(\sqrt{x^2+x+1}+2x+1\right)}{2\cdot\sqrt{x^2+x+1}\left(\sqrt{x^2+x+1}+\frac{x}{2}+1\right)} \cdot \frac{\sqrt{x^2+x+1}-\frac{x}{2}-1}{\sqrt{x^2+x+1}-\frac{x}{2}-1} = \frac{-\left[x^2+x+1-x^2-\frac{5}{2}x-1+\frac{3}{2}x\sqrt{x^2+x+1}\right]}{2\cdot\sqrt{x^2+x+1}\left(x^2+x+1-\frac{x^2}{4}-x-1\right)} =$$

$$= \frac{-\left[-\frac{3}{2}x+\frac{3}{2}x\sqrt{x^2+x+1}\right]}{2\cdot\sqrt{x^2+x+1}\cdot\frac{3}{4}x^2} = \frac{\overset{1}{\cancel{2x}}-\cancel{2x}\sqrt{x^2+x+1}}{x\sqrt{x^2+x+1}} = \frac{\overset{1}{\cancel{x}}-\cancel{x}\sqrt{x^2+x+1}}{x\sqrt{x^2+x+1}} = \frac{1-\sqrt{x^2+x+1}}{x\sqrt{x^2+x+1}}$$

4) $$\frac{1+\frac{x+\frac{1}{2}}{\sqrt{x^2+x+1}}}{2\left(\sqrt{x^2+x+1}+x+\frac{1}{2}\right)} = \frac{\sqrt{x^2+x+1}+x+\frac{1}{2}}{2\sqrt{x^2+x+1}\left(\sqrt{x^2+x+1}+x+\frac{1}{2}\right)} = \frac{1}{2\sqrt{x^2+x+1}}$$

1) + 2) + 3) + 4):

$$\frac{x+\frac{1}{2}}{\sqrt{x^2+x+1}} + \frac{1}{x} + \frac{\overset{1}{\cancel{2x}}-\cancel{2x}\sqrt{x^2+x+1}}{x\sqrt{x^2+x+1}} + \frac{1}{2\sqrt{x^2+x+1}}$$

$$= \frac{x(x+\frac{1}{2})+\sqrt{x^2+x+1}+\overset{1}{\cancel{2x}}-\cancel{2x}\sqrt{x^2+x+1}+\frac{x}{2}}{x\sqrt{x^2+x+1}}$$

$$= \frac{x(x+\frac{1}{2})+\sqrt{x^2+x+1}+1-\sqrt{x^2+x+1}+\frac{x}{2}}{x\sqrt{x^2+x+1}}$$

$$= \frac{x^2+x+1}{x\sqrt{x^2+x+1}} = \frac{\sqrt{x^2+x+1}}{x}$$

17. $\int \sqrt{x^2+2x-3}\,dx$

$$x^2+2x-3 = x^2+2x+1-4 = (x+3)(x-1)$$

DO: $x \in (-\infty; -3) \cup (1; +\infty)$

Eulerova substituce:

$$\sqrt{x^2+2x-3} = x+u$$

$$x^2+2x-3 = x^2+2xu+u^2$$

$$2x(1-u) = u^2+3; \qquad x = \frac{u^2+3}{-2(u-1)}$$

$$\begin{array}{l|l} u^2+3 & u-1 \\ \hline -(u^2-u) & u+1 \\ \quad u+3 & \\ \; -(u-1) & \\ \hline \quad 4 & \end{array}$$

$$x = \frac{u^2+3}{-2(u-1)} = \frac{-1}{2}\left[u+1+\frac{4}{u-1}\right]$$

$$dx = -\frac{1}{2}\left[1-\frac{4}{(u-1)^2}\right]du$$

$$= -\frac{(u-1)^2-4}{2(u-1)^2}\,du = \frac{(u-3)(u+1)}{-2(u-1)^2}\,du$$

$$u = \sqrt{x^2+2x-3}-x = \frac{2x-3}{\sqrt{x^2+2x-3}+x}$$

$x \to +\infty: \; u \to 1$

$x \to 1 \; : \; u \to -1$

$x \to -\infty: \; u \to +\infty$

$x \to -3: \; u \to 3$

$x \in (-\infty,-3) \Leftrightarrow u \in (3,+\infty)$

$x \in (1,+\infty) \Leftrightarrow u \in (-1,1)$

$$\sqrt{x^2+2x-3} = u+x = u-\frac{1}{2}(u+1)-\frac{2}{u-1} =$$

$$= \frac{u^2-u-\frac{1}{2}(u^2-1)-2}{u-1} = \frac{\frac{1}{2}u^2-u-\frac{3}{2}}{u-1} = \frac{u^2-2u-3}{2(u-1)} = \frac{(u-3)(u+1)}{2(u-1)}$$

$$I = \int\sqrt{x^2+2x-3}\,dx = \int \frac{u^2-2u-3}{2(u-1)}\cdot\frac{(u-3)(u+1)}{-2(u-1)^2}\,du = \frac{-1}{4}\int\frac{(u^2-2u-3)^2\,du}{(u-1)^3}$$

$$= \left|\begin{matrix} v=u-1 \\ u=v+1 \end{matrix}\right| = \frac{-1}{4}\int\frac{(v-2)^2(v+2)^2}{v^3}\,dv = \frac{-1}{4}\int\frac{(v^2-4)^2}{v^3}\,dv = \frac{-1}{4}\int\frac{v^4-8v^2+16}{v^3}\,dv$$

$$= -\frac{1}{4}\int\left(v-\frac{8}{v}+\frac{16}{v^3}\right)dv = -\frac{v^2}{8}+2\log|v|+\frac{2}{v^2}+C$$

- $v = u-1 = \sqrt{x^2+2x-3}-x-1$

$$\frac{1}{v} = \frac{1}{\sqrt{x^2+2x-3}-x-1}\cdot\frac{\sqrt{x^2+2x-3}+x+1}{\sqrt{x^2+2x-3}+x+1} = \frac{\sqrt{x^2+2x-3}+x+1}{(x^2+2x-3)-(x+1)^2} = \frac{-1}{4}\left(\sqrt{x^2+2x-3}+x+1\right)$$

$$\frac{2}{v^2} = \frac{1}{8}\left(\sqrt{x^2+2x-3}+x+1\right)^2 = \frac{1}{8}\left(2x^2+4x-2+2(x+1)\sqrt{x^2+2x-3}\right) = \frac{x^2+2x-1+(x+1)\sqrt{x^2+2x-3}}{4}$$

- $$-\frac{v^2}{8} = \frac{-1}{8}\left[2x^2+4x-2-2(x+1)\sqrt{x^2+2x-3}\right] = \frac{-x^2-2x+1+(x+1)\sqrt{x^2+2x-3}}{4}$$

- $\frac{2}{v^2} - \frac{v^2}{8} = \frac{1}{2}(x+1)\sqrt{x^2+2x-3}$

$$I = \frac{1}{2}(x+1)\sqrt{x^2+2x-3} - 2\log\left|\sqrt{x^2+2x-3}+x+1\right| + C$$

$C = C_1$ pro $x \in (-\infty, -3)$

$C = C_2$ pro $x \in (1, +\infty)$

Kontrola:

1) $$\frac{1}{2}\sqrt{x^2+2x-3} + \frac{1}{2}(x+1)\frac{(x+1)}{\sqrt{x^2+2x-3}} = \frac{x^2+2x-3+(x+1)^2}{2\sqrt{x^2+2x-3}} = \frac{2x^2+4x-2}{2\sqrt{x^2+2x-3}} = \frac{x^2+2x-1}{\sqrt{x^2+2x-3}}$$

2) $$\frac{-2\left(\frac{x+1}{\sqrt{x^2+2x-3}}+1\right)}{\sqrt{x^2+2x-3}+x+1} = \frac{-2}{\sqrt{x^2+2x-3}}$$

1) + 2) : $$\frac{x^2+2x-1-2}{\sqrt{x^2+2x-3}} = \sqrt{x^2+2x-3}$$